



A bifurcation method for solving multiple positive solutions to the boundary value problem of the Henon equation on a unit disk

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ABSTRACT

Three algorithms based on the bifurcation theory are proposed to compute the $O(2)$ symmetric positive solutions to the boundary value problem of the Henon equation on the unit disk. Taking l in the Henon equation as a bifurcation parameter, the symmetry-breaking bifurcation point on the branch of the $O(2)$ symmetric positive solutions is found via the extended systems. Finally, other symmetric positive solutions are computed by the branch switching method based on the Lyapunov–Schmidt reduction.

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1. Introduction

We consider the boundary value problem (BVP) of the Henon equation

$$g(u, l) = \begin{cases} \Delta u + |\mathbf{x}|^l |u|^{q-1} u = 0, & \mathbf{x} \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (1.1)$$

where Δ is the Laplace operator, $l \geq 0$, $\mathbf{x} = (x_1, x_2)$, Ω is the unit disk in \mathbb{R}^2 , $\partial\Omega$ is the boundary of Ω and $q \geq \frac{1}{2}$, which is called the polytropic index in astrophysics. The Henon equation (1.1) was proposed by Henon in [1] when he studied the stability of the rotating stellar structures and up to now the properties of the solutions to the Henon equation (1.1) has been studied by many authors [2–6].

Since 60s of the 20th century, the existence and multiplicity of solutions to the boundary value problems of the nonlinear elliptic PDE's such as problem (1.1) are studied by the monotone iterative method in the ordered Banach space [7,8] and the mountain pass lemma and the min–max theorem in the critical point theory [9,10]. It becomes an important field in PDE study. But what distribution and structure the solutions to the BVP of the nonlinear elliptic equations have and how to compute them have attracted the attention of many mathematicians, physicists and engineers.

There are mainly five numerical methods for computing such kinds of problems: the Monotone Iterative Scheme (MIS) [11,12], the Mountain Pass Algorithm (MPA) [13], the High Linking Algorithm (HLA) [14], the Min–Max Algorithm (MMA) [15,16] and the Search Extension Method (SEM) [17]. MIS is based on the monotone iterative methods in the ordered Banach space. MPA, MMA and HLA are based on the numerical implementation of the mountain pass lemma and the

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min–max theorem in the critical point theory. MPA was proposed by Choi and McKenna to compute the solutions with the Morse Index (MI) 0 or 1. Ding, Costa and Chen established HLA for sign-changing solution (MI = 2) of semilinear elliptic problems. Li and Zhou designed a new min–max algorithm (MMA) to find multiple saddle points with any Morse index which is more constructive than the traditional min–max theorem. Chen and Xie proposed SEM, which search to the initial guess based the linear combination of the eigenfunctions of the linearized problem and then get the better initial guess by the continuation method for the discretized problem by the finite element method. The bifurcation method [18–20] is applied successfully to solving the BVP of the Henon equation on the unit square. The advantages of the bifurcation method are that it can compute the solutions to problem (1.1) with any Morse index and the different symmetries as many as possible and it can simplify the computation of problem (1.1). On the other hand, we can overcome the difficulty in searching the initial guess in other methods by using the bifurcation method.

We embed (1.1) to the nonlinear bifurcation problems with parameter λ of the following form:

$$G(u, \lambda, l) = \begin{cases} \Delta u + \lambda u + |\mathbf{x}|^l |u|^{q-1} u = 0, & \mathbf{x} \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (1.2)$$

where $\lambda \in \mathbb{R}$. According to the bifurcation theory [21,22], (1.2) has nontrivial solution branches bifurcated from the trivial solution near the bifurcation points. Along the nontrivial solution branches we can get the solutions to problems (1.1) by the continuation method when the parameter λ goes to 0. The most interesting solutions to problems (1.1) are the positive solutions with $O(2)$ -symmetry or other symmetry.

In Section 2, we get the approximate expression of the nontrivial solutions of problem (1.2) and bifurcation equation of problem (1.2) near the bifurcation point via the Lyapunov–Schmidt reduction. In Section 3, we discrete problem (1.2) by using the central difference scheme and get D_M equivariance for the discretized problem. In Section 4, three algorithms are proposed to compute the $O(2)$ symmetric positive solutions of problem (1.1), based on the bifurcation theory. In Section 5, we propose the extended systems, which can detect the symmetry-breaking bifurcation points on the branch of the $O(2)$ symmetric positive solutions. In Section 6, other symmetric positive solutions are computed by the branch switching method based on the Lyapunov–Schmidt reduction. Finally, in Section 7 the numerical results are given and the multiple positive solutions to problem (1.1) are visualized.

2. The Lyapunov–Schmidt reduction of a nonlinear bifurcation problem

Using polar coordinates, problem (1.2) may be transformed to

$$G(u, \lambda, l) = \begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \lambda u + r^l |u|^{q-1} u = 0, & r \in [0, 1], \theta \in [0, 2\pi], \\ u|_{r=1} = 0, \end{cases} \quad (2.1)$$

where $r = \sqrt{x_1^2 + x_2^2}$, $\tan \theta = x_2/x_1$, $r \in [0, 1]$, $\theta \in [0, 2\pi]$.

Obviously, the problem (1.2) is $\Gamma = O(2) \times Z_2$ equivariant, namely

$$G(\gamma u, \lambda) = \gamma G(u, \lambda), \quad \forall \gamma \in \Gamma = O(2) \times Z_2,$$

where $O(2) = \{I, S, R_\alpha (\alpha \in [0, 2\pi])\}$, $R_\alpha u(r, \theta) = u(r, \theta + \alpha)$, $Su(r, \theta) = u(r, -\theta)$, $Z_2 = \{I, -I\}$. For $\forall \lambda \in \mathbb{R}$, $u \equiv 0$ is a Γ symmetric trivial solution of (1.2). We consider the $O(2)$ symmetric nontrivial solution $u(r, \theta)$, which is independent of θ , $u(r)$ should satisfy the following two-point boundary value problem

$$g(u, \lambda, l) = \begin{cases} \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \lambda u + r^l |u|^{q-1} u = 0, & r \in [0, 1], \\ u(1) = 0, \quad u'(0) = 0. \end{cases} \quad (2.2)$$

Problem (2.2) has nontrivial solution branches bifurcated from the trivial solution near the bifurcation points. Along the nontrivial solution branches we can get the $O(2)$ symmetric solutions to problems (1.1) with the continuation method when the parameter λ goes to 0, which satisfy

$$\begin{cases} \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \lambda u + r^l |u|^{q-1} u = 0, & r \in [0, 1], \\ u(1) = 0, \quad u'(0) = 0. \end{cases} \quad (2.3)$$

The linearized problem of problem (2.2) at the trivial solution

$$\begin{cases} \frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} + \lambda \phi = 0, & r \in [0, 1], \\ \phi(1) = 0, \quad \phi'(0) = 0 \end{cases} \quad (2.4)$$

has the eigenvalues $\lambda_n = \mu_n^2$ ($n = 1, 2, \dots$) and the corresponding eigenfunction $\phi_n = J_0(\mu_n r)$, where μ_n is the n th positive zero point of $J_0(r)$, which is the Bessel function of order 0: $J_0(r) = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2^{2k}(k!)^2} (r)^{2k} \right)$. An inner product is defined by

$$\langle u, v \rangle = \int_0^1 u v dr.$$

Let

$$L_n = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \lambda_n,$$

$$Y = \{u | u \in C^0[0, 1]\}, \quad X = \{u | u \in C^2[0, 1], u(1) = 0, u'(0) = 0\}.$$

$L_n : X \rightarrow Y$ is a Fredholm operator with index zero. Space X and Y have the decomposition:

$$X = \text{Ker } L_n \oplus M, \quad Y = N^* \oplus R(L_n),$$

where $\text{Ker } L_n = \{\alpha \phi_n | \alpha \in \mathbb{R}, \phi_n = J_0(\mu_n r)\}$, $M = (\text{Ker } L_n)^\perp \cap X$, $N^* = (R(L_n))^\perp = \text{Ker } L_n^*$, L_n^* is the conjugate operator of L_n , $L_n^* = \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \lambda_n$.

Let $\text{Ker } L_n^* = \{\alpha \psi_n(r) | \alpha \in \mathbb{R}\}$, method of power series expansion can lead

$$\psi_n(r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k+1)!(k!)} (\mu_n r)^{2(k+1)}.$$

Let P be a map from Y to $R(L_n)$,

$$Pz = z - (z, e^*)e^*, \quad z \in Y,$$

where $e^* = \frac{\psi_n}{\|\psi_n\|}$. By the Lyapunov–Schmidt reduction, problem (2.2) is equivalent to

$$Pg(\tau \phi_n + w, \lambda_n + \eta, l) = 0, \quad \tau, \eta \in \mathbb{R}, w \in M, \quad (2.5)$$

$$\langle e^*, g(\tau \phi_n + w, \lambda_n + \eta, l) \rangle = 0, \quad (2.6)$$

where $u = \tau \phi_n + w$, $w \in W$, $\lambda = \lambda_n + \eta$. Since $Pg_u(0, \lambda_n, l) = PL_n = L_n$, and L_n restricted in M is regular, Eq. (2.5) has unique solution $w = w(\tau, \mu)$, which satisfies $w(0, 0) = 0$ by the implicit function theorem near $(\tau, \mu) = (0, 0)$. Substituting $w(\tau, \mu)$ into (2.6) yields

$$f(\tau, \eta) = \langle e^*, g(\tau \phi_n + w(\tau, \eta), \lambda_n + \eta, l) \rangle = \left\langle e^*, \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} + \lambda_n w + h(\tau, \eta, w) \right\rangle = 0, \quad (2.7)$$

where $h(\tau, \eta, w) = \eta(\tau \phi_n + w) + r^l |\tau \phi_n + w|^{(q-1)} (\tau \phi_n + w)$. Since $\langle e^*, L_n w \rangle = \langle L_n^* e^*, w \rangle = 0$, the Eq. (2.7) can be simplified to

$$f(\tau, \eta) = \langle e^*, h(\tau, \eta, w) \rangle = 0. \quad (2.8)$$

In the following, we calculate the approximate expressions of $w(\tau, \mu)$ and $f(\tau, \mu)$ for $q = 3$. Differentiating (2.5) with respect to τ we have

$$P(g_u(\tau \phi_n + w, \lambda_n + \eta, l)(\phi_n + w_\tau)) = 0,$$

which evaluated at $(0, 0)$ leads to

$$L_n w_\tau(0, 0) = 0.$$

It follows that $w_\tau(0, 0) = 0$, due to $w_\tau(0, 0) \in M$, and L_n is invertible in M .

From (2.8) we get

$$f_\tau(\tau, \eta) = \langle e^*, \eta(\phi_n + w_\tau) + 3r^l (\tau \phi_n + w)^2 (\phi_n + w_\tau) \rangle,$$

which evaluated at $(0, 0)$ leads to

$$f_\tau(0, 0) = 0.$$

Similarly, we get

$$\begin{aligned} w_{\eta^k}(0, 0) &= 0, \quad \forall k = 1, 2, \dots, & f_{\eta^k}(0, 0) &= 0, \quad \forall k = 1, 2, \dots, \\ w_{\tau^2}(0, 0) &= 0, & f_{\tau^2}(0, 0) &= 0, \\ w_{\tau^3}(0, 0) &= -6L_n^{-1}(P(r^l \phi_n^3)), & f_{\tau^3}(0, 0) &= \langle e^*, 6r^l \phi_n^3 \rangle, \\ w_{\tau\eta}(0, 0) &= 0, & f_{\tau\eta}(0, 0) &= \langle e^*, \phi_n \rangle = c_n \neq 0. \end{aligned}$$

Therefore we have approximately

$$u(\tau, \eta) \doteq \tau \phi_n - \tau^3 L_n^{-1}(P(r^l \phi_n^3)) \quad (2.9)$$

and

$$f(\tau, \eta) \doteq c_n \tau \eta + \langle e^*, r^l \phi_n^3 \rangle \tau^3 = 0. \quad (2.10)$$

3. The equivariance of difference equations

The discretized equations of problem (2.1) with central difference are

$$G_h(u_h, \lambda, l) = \begin{cases} \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\delta r)^2} + \frac{u_{i+1,j} - u_{i-1,j}}{2r_i \delta r} & i = 1, \dots, N-1, \\ + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(r_i)^2 (\delta \theta)^2} + \lambda u_{i,j} + r_i^l |u_{i,j}|^{q-1} u_{i,j} = 0, & j = 1, \dots, M, \\ u_{N,j} = 0, & j = 1, \dots, M, \\ \frac{4}{M} \left(\sum_{j=1}^M u_{1,j} - M u_{0,0} \right) + \lambda (\delta r)^2 u_{0,0} + \delta_{0,l} (\delta r)^2 |u_{0,0}|^{q-1} u_{0,0} = 0, \end{cases} \quad (3.1)$$

where

$$\delta r = \frac{1}{N}, \quad \delta \theta = \frac{2\pi}{M}, \quad r_i = i\delta r, \quad \theta_j = j\delta \theta, \quad u_h = \{u_{ij}, i = 1, 2, \dots, N-1, j = 1, 2, \dots, M\},$$

$$u_{i,j} = u(r_i, \theta_j), \quad \delta_{0,l} = \begin{cases} 1, & l = 0, \\ 0, & l \neq 0. \end{cases}$$

The problem (3.1) is Γ equivariant, namely

$$\begin{aligned} G_h(\gamma u_h, \lambda, l) &= \gamma G_h(u_h, \lambda, l), \quad \forall \gamma \in \Gamma = D_M \times Z_2, \\ D_M &= \{I, R_1, \dots, R_{M-1}, S_1, S_2, \dots, S_M\}, & Ru_{ij} &= u_{i,j+1}, \\ R_k &= R^k, & Su_{ij} &= u_{i,M-j}, \\ S_k &= R_{k-1}S, & Z_2 &= \{I, -I\}. \end{aligned}$$

The linearized problem of problem (2.1) at the trivial solution is

$$\begin{cases} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \lambda \phi = 0, & 0 \leq r < 1, \theta \in [0, 2\pi], \\ \phi|_{r=1} = 0. \end{cases} \quad (3.2)$$

The discretized equations of (3.2) are

$$\begin{cases} \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(\delta r)^2} + \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2r_i \delta r} & i = 1, \dots, N-1, \\ + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{(r_i)^2 (\delta \theta)^2} + \lambda \phi_{i,j} = 0, & j = 1, \dots, M, \\ \phi_{N,j} = 0, & j = 1, \dots, M, \\ \frac{4}{M} \left(\sum_{j=1}^M \phi_{1,j} - M \phi_{0,0} \right) + \lambda (\delta r)^2 \phi_{0,0} = 0, \end{cases} \quad (3.3)$$

where $\phi_{i,j} = \phi(r_i, \theta_j)$, $r_i = i\delta r$, $\theta_j = j\delta \theta$.

4. Computation of $O(2)$ symmetric positive solutions to (1.1)

The first algorithm:

Step 1: Calculate the first eigenvalue λ_0 and the corresponding eigenfunction ϕ_0 of the discretized problem (3.3), where ϕ_0 is D_M symmetric.

Step 2: Let $u_h = \tau \phi_0 + w_h$, $\eta = \lambda - \lambda_0$, where $\tau \in \mathbb{R}$ is a parameter, w_h, ϕ_0 with D_M symmetry and $\eta \in \mathbb{R}$. The Lyapunov–Schmidt reduction leads to

$$\begin{cases} \frac{\partial^2 w_h}{\partial r^2} + \frac{1}{r} \frac{\partial w_h}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w_h}{\partial \theta^2} + (\eta + \lambda_0) w_h + \eta \tau \phi_0 \\ + r^l |\tau \phi_0 + w_h|^{q-1} (\tau \phi_0 + w_h) = 0, & r \in [0, 1], \theta \in [0, 2\pi], \\ w_h|_{r=1} = 0, \\ \langle \phi_0, w_h \rangle = 0. \end{cases} \quad (4.1)$$

Table 1

The initial guess and the number of Newton iteration for solving problem (4.5) directly.

l	0	2	4	6	8	10
τ_0	3.9511	8.5160	15.0518	23.6528	34.5237	47.7959
The number of Newton iteration	4	4	4	6	6	7

Its discretized equations are

$$\begin{cases} \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{(\delta r)^2} + \frac{w_{i+1,j} - w_{i-1,j}}{2r_i \delta r} + \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{(r_i)^2 (\delta \theta)^2} & i = 1, \dots, N-1, \\ + (\eta + \lambda_0) w_{i,j} + \eta \tau \phi_{i,j} + r_i^l |\tau \phi_{i,j} + w_{i,j}|^{q-1} (\tau \phi_{i,j} + w_{i,j}) = 0, & j = 1, \dots, M, \\ w_{N,j} = 0, & j = 1, \dots, M, \\ \frac{4}{M} \left(\sum_{j=1}^M w_{1,j} - M w_{0,0} \right) + (\delta r)^2 ((\eta + \lambda_0) w_{0,0} + \eta \tau \phi_{0,0} + \delta_{0,l} |\tau \phi_{0,0} + w_{0,0}|^{q-1} (\tau \phi_{0,0} + w_{0,0})) = 0, \\ \phi_{0,0} w_{0,0} + \sum_{i=1}^{N-1} \sum_{j=1}^M \phi_{i,j} w_{i,j} = 0, \end{cases} \quad (4.2)$$

where $w_{i,j} = w(r_i, \theta_j)$, $\phi_{i,j} = \phi_0(r_i, \theta_j)$, $r_i = i\delta r$, $\theta_j = j\delta\theta$. The Newton iteration method is used to solve problem (4.2). When τ is continued from 0.1 to τ_{end} , we get the solution $(w_h)_{end}$ and η_{end} which are far away from 0.

Step 3: Taking $u_{end} = \tau_{end}\phi_0 + w_{end}$, $\lambda_{end} = \lambda_0 + \eta_{end}$ as a starting point, the continuation method for λ and the Newton iteration method are used to solve problem (3.1). Finally, we get the D_M symmetric positive solution u to problem (2.1), when λ is continued from λ_{end} to 0.

The second algorithm:

Taking $\eta = -\lambda_1 = -\mu_1^2$, we can get a solution to Eq. (2.10)

$$\tau_0 = \sqrt{\frac{\langle \psi_1, \phi_1 \rangle}{\langle \psi_1, r^l \phi_1^3 \rangle}} \mu_1. \quad (4.3)$$

We can take

$$u_0 = \tau_0 \phi_1$$

or

$$u_0 = \tau_0 \phi_1 - \tau_0^3 L_1^{-1}(P(r^l \phi_1^3))$$

as the initial guess of the Newton iteration for solving problem (2.3) directly. Here, $L_1^{-1}(P(r^l \phi_1^3))$ may be got through numerical computation of the following boundary value problem:

$$\begin{cases} \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} + \lambda_1 w + P(r^l \phi_1^3) = 0, \\ w(1) = w'(0) = 0, \\ \langle \phi_1, w \rangle = 0. \end{cases} \quad (4.4)$$

The discretized equations of problem (2.3) are

$$\begin{cases} \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{u_{i+1} - u_{i-1}}{2ih^2} + (ih)^l |u_i|^{q-1} u_i = 0, & i = 1, 2, \dots, N-1, \\ u_N = 0, \\ 3u_0 - 4u_1 + u_2 = 0, \end{cases} \quad (4.5)$$

where $[0, 1]$ is divided into N small interval, $N = 500$, $u_i = u(r_i) = u(ih)$. Table 1 shows the number of Newton iteration for solving problem (4.5) directly, when we take $u^0(r) = \tau_0 J_0(\mu_1 r)$ as the initial guess of the Newton iteration.

The third algorithm: Taking l as a parameter, the $O(2)$ symmetric positive solutions to problem (1.1) can be computed with continuation.

The $O(2)$ symmetric positive solution to problem (1.1) when $l = 0$ (the Lane–Emden equation) is used as a starting point on the $O(2)$ symmetric positive solution branch Γ to problem (1.1) with varied l which can be computed effectively by the l continuation and the Newton iteration method. While l is continued, the eigenvalues of Jacobian $[g(u, l)]_u$ are monitored. The eigenvalues with small absolute value are found for l near 0.58, 3.19, 5.78, 8.35, 10.90, 13.43 and 15.94. The corresponding eigenvectors have Σ_1 , Σ_d , D_3 , D_4 , D_5 , D_6 and D_7 symmetry respectively. These are the potential symmetry-breaking bifurcation points [21,22].

5. Computation of the symmetry-breaking bifurcation point on the $O(2)$ symmetric positive solution branch

In the following, let Σ be one of $\Sigma_1, \Sigma_d, D_3D_4, D_5, D_6$ and D_7, X^Σ be the invariant subspace of Σ . Since

$$g(\gamma u, l) = \gamma g(u, l), \quad \forall \gamma \in O(2), \quad (5.1)$$

X^Σ can be decomposed into $X^\Sigma = X^{O(2)} \oplus W$, where $W = X^\Sigma \cap (X^{O(2)})^\perp$, $(X^{O(2)})^\perp$ is an orthogonal complement of $X^{O(2)}$.

Lemma 1. If $u \in X^{O(2)}$, then the following conclusions hold $\forall l \in \mathbb{R}$:

- (i) $g(u, l), g_l(u, l) \in X^{O(2)}$,
- (ii) $X^{O(2)}$ and W are the invariant subspaces of $g_u(u, l)$ and $g_{ul}(u, l)$,
- (iii) if $v \in X^{O(2)}$, then $X^{O(2)}$ and W are the invariant subspaces of $g_{uu}(u, l)v$.

Proof. (i) From equivariance of (5.1), if $u \in X^{O(2)}$,

$$g(u, l) = \gamma g(u, l), \quad \forall \gamma \in O(2),$$

$$g_l(u, l) = \gamma g_l(u, l), \quad \forall \gamma \in O(2),$$

which means $g(u, l), g_l(u, l) \in X^{O(2)}$.

(ii)

(1) From (5.1), it is known that

$$\gamma g_u(u, l) = g_u(\gamma u, l)\gamma, \quad \forall \gamma \in O(2). \quad (5.2)$$

If $u, v \in X^{O(2)}$, we get

$$\gamma g_u(u, l)v = g_u(u, l)v, \quad \forall \gamma \in O(2),$$

which means $g_u(u, l)v \in X^{O(2)}$, $X^{O(2)}$ is an invariant subspace of $g_u(u, l)$. Similarly, $X^{O(2)}$ is an invariant subspace of $g_{ul}(u, l)$, X^Σ also is an invariant subspace of $g_u(u, l)$ and $g_{ul}(u, l)$.

(2) On the other hand, from (5.2) we can know

$$g_u^\top(u, l)\gamma^\top = \gamma^\top g_u^\top(\gamma u, l), \quad \forall \gamma \in O(2). \quad (5.3)$$

If $v \in X^{O(2)}$, then

$$g_u^\top(u, l)v = \gamma^\top g_u^\top(u, l)v, \quad \forall \gamma \in O(2),$$

which means $g_u^\top(u, l)v \in X^{O(2)}$, namely, $X^{O(2)}$ is an invariant subspace of $g_u^\top(u, l)$. Similarly, $X^{O(2)}$ is an invariant subspace of $g_{ul}^\top(u, l)$, X^Σ also is an invariant subspace of $g_u^\top(u, l)$ and $g_{ul}^\top(u, l)$.

(3) If $v \in X^{O(2)}$, then $g_{uu}^\top(u, l)v \in X^{O(2)}$,

$$\langle g_u^\top(u, l)v, w \rangle = 0, \quad \forall w \in W, v \in X^{O(2)},$$

$$\langle v, g_u(u, l)w \rangle = 0, \quad \forall w \in W, v \in X^{O(2)},$$

therefore $g_u(u, l)w \in W$, namely, W is an invariant subspace of $g_u(u, l)$. Similarly, W is an invariant subspace of $g_{ul}(u, l)$.

(iii) From (5.1), it is true that

$$\gamma g_{uu}(u, l)vw = g_{uu}(\gamma u, l)\gamma v\gamma w, \quad \forall \gamma \in O(2).$$

If $u, v, w \in X^{O(2)}$, we have

$$\gamma g_{uu}(u, l)vw = g_{uu}(u, l)vw, \quad \forall \gamma \in O(2),$$

which means $g_{uu}(u, l)vw \in X^{O(2)}$, namely $X^{O(2)}$ is an invariant subspace of $g_{uu}(u, l)v$. Similarly X^Σ also is an invariant subspace of $g_{uu}(u, l)v$.

Also, we have

$$\gamma g_{uu}(u, l)v = g_{uu}(\gamma u, l)\gamma v\gamma.$$

Therefore

$$(g_{uu}(u, l)v)^\top \gamma^\top = \gamma^\top (g_{uu}(\gamma u, l)\gamma v)^\top,$$

$$(g_{uu}(u, l)v)^\top \gamma^\top w = \gamma^\top (g_{uu}(\gamma u, l)\gamma v)^\top w.$$

If $u, v, w \in X^{O(2)}$, then $(g_{uu}(u, l)v)^\top w \in X^{O(2)}$, which means $X^{O(2)}$ is an invariant subspace of $(g_{uu}(u, l)v)^\top$. If $w \in W, v, h \in X^{O(2)}$, then

$$\langle h, g_{uu}vw \rangle = \langle (g_{uu}v)^\top h, w \rangle = 0,$$

which means $g_{uu}vw \in W$, thus W is an invariant subspace of $g_{uu}v$. \square

Table 2
Symmetry-breaking bifurcation point.

	$O(2) - \Sigma_1$	$O(2) - \Sigma_d$	$O(2) - D_3$	$O(2) - D_4$	$O(2) - D_5$	$O(2) - D_6$	$O(2) - D_7$
l	0.5988	3.1935	5.7786	8.3499	10.9031	13.4339	15.9381

On the $O(2)$ symmetric positive solution branch, if there is a point (u_0, l_0) , at which $g_u^0 = g_u(u_0, l_0)$ is singular and its null space is $N(g_u^0) = \text{span}\{\phi_0\}$, its range space is $R(g_u^0) = \{x \in X^\Sigma | \langle \psi_0, x \rangle = 0\}$, where $\phi_0 \in W$ and $\psi_0 \in W$ are null eigenvector of g_u^0 and $(g_{ul}^0)^T$ respectively, and if

$$\langle \psi_0, (g_{uu}^0 v_l + g_{ul}^0) \phi_0 \rangle \neq 0, \quad (5.4)$$

where $v_l \in X^{O(2)}$ is the unique solution to

$$g_u^0 v_l + g_l^0 = 0, \quad (5.5)$$

then the point (u_0, l_0) on the $O(2)$ symmetric positive solution branch of problem (1.1) is called $O(2) - \Sigma$ symmetry-breaking bifurcation point with respect to l .

The following is the extended system for detecting the $O(2) - \Sigma$ symmetry-breaking bifurcation points:

$$G(y) = \begin{pmatrix} g(u, l) \\ g_u(u, l)\phi \\ \langle h_0, \phi \rangle - 1 \end{pmatrix} = 0, \quad (5.6)$$

where $y = (u, \phi, l) \in Y = X^{O(2)} \times W \times \mathbb{R}$, $y_0 = (u_0, \phi_0, l_0)$, $h_0 \in W$ is a normalization of ϕ_0 .

Theorem 1. The extended system (5.6) is regular at the $O(2) - \Sigma$ symmetry-breaking bifurcation point $y_0 = (u_0, \phi_0, l_0)$.

Proof. Obviously,

$$G_y^0 = \begin{pmatrix} g_u^0 & 0 & g_l^0 \\ g_{uu}^0 \phi_0 & g_u^0 & g_{ul}^0 \phi_0 \\ 0 & \langle h_0, \cdot \rangle & 0 \end{pmatrix}.$$

Next, we prove that $G_y^0 : Y \rightarrow Y$ is one-to-one. Let

$$G_y^0 Z = 0, \quad (5.7)$$

where $Z = (v, w, \alpha)^T$, $v \in X^{O(2)}$, $w \in W$, $\alpha \in \mathbb{R}$. Expanding (5.7) yields

$$g_u^0 v + \alpha g_l^0 = 0, \quad (5.8)$$

$$g_{uu}^0 \phi_0 v + g_u^0 w + \alpha g_{ul}^0 \phi_0 = 0, \quad (5.9)$$

$$\langle h_0, w \rangle = 0. \quad (5.10)$$

From (5.8), we can get $v = \alpha v_l$. Substituting $v = \alpha v_l$ into (5.9) and taking an inner product with ψ_0 lead to $\alpha \langle \psi_0, (g_{uu}^0 v_l \phi_0 + g_{ul}^0 \phi_0) \rangle = 0$. Therefore $\alpha = 0$ due to (5.4). Also $w = 0$ by solving $g_u^0 w = 0$, $\langle h_0, w \rangle = 0$. Similarly we can prove that $G_y^0 : Y \rightarrow Y$ is onto. Therefore G_y^0 is regular. \square

Since G_y^0 is regular, we can solve the extended system (5.6) with Newton iteration method. During continuation of $O(2)$ symmetric positive solutions, we can find some $u^* \in X^{O(2)}$, $l^* \in \mathbb{R}$, at which the Jacobian $g_u(u, l)$ has eigenvalues with small absolute value. (u^*, l^*) and the corresponding eigenvector of the eigenvalue with small absolute value can be used as the initial guess for the Newton iteration. The numerical results are given in Table 2.

6. Branch switching to Σ symmetric solutions

Let the $O(2) - \Sigma$ symmetry-breaking bifurcation point be $l = l_0$, $u = u_0 \in X^{O(2)}$, $\psi_0 \in W$, $\phi_0 \in W$. The numerical computation shows

$$a = \langle \psi_0, g_{uu}^0 \phi_0 \phi_0 \rangle = 0,$$

$$b = \langle \psi_0, (g_{uu}^0 v_l + g_{ul}^0) \phi_0 \rangle \neq 0,$$

$$c = \langle \psi_0, (g_{uu}^0 v_l v_l + 2g_{ul}^0 v_l + g_{ll}^0) \rangle = 0,$$

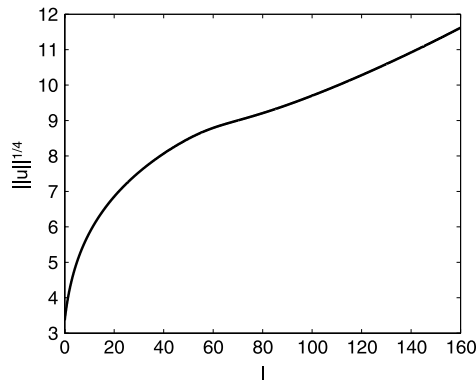


Fig. 1. $O(2)$ symmetric positive solutions branch.

where $v_l \in X^{O(2)}$ is the unique solution to Eq. (5.5). Therefore assumption (5.4) is satisfied. From the algebraic bifurcation equation [21,22], we know that at the symmetry-breaking bifurcation point, the tangent vector along the $O(2)$ symmetric positive solution branch is $(v_l, 1)$, the tangent vector along the Σ symmetric positive solution branch is $(\phi_0, 0)$. We define that

$$F(w, \eta, \varepsilon) = \begin{cases} \frac{1}{\varepsilon} g(u_s(l_0 + \eta) + \varepsilon(\phi_0 + w), l_0 + \eta), & \varepsilon \neq 0, \\ g_u(u_s(l_0 + \eta), l_0 + \eta)(\phi_0 + w), & \varepsilon = 0, \end{cases} \quad (6.1)$$

$$N(w, \eta, \varepsilon) = \langle \phi_0, w \rangle, \quad (6.2)$$

where $u_s(l_0 + \eta)$ are the $O(2)$ symmetric positive solutions, $w \in W$, $\eta, \varepsilon \in \mathbb{R}$. Obviously, $F(0, 0, 0) = 0$, $N(0, 0, 0) = 0$. Jacobian of (6.1), (6.2) with respect to w, η at $(w, \eta, \varepsilon) = (0, 0, 0)$ is that

$$A^0 = \frac{\partial(F, N)}{\partial(w, \eta)}|_{(0,0,0)} = \begin{pmatrix} g_u^0 & B^0 \\ \langle \phi_0, \cdot \rangle & 0 \end{pmatrix},$$

where $B^0 = [g_{uu}^0(u_s(l_0), l_0)u_s'(l_0) + g_{ul}^0(u_s(l_0), l_0)]\phi_0 = [g_{uu}^0 v_l + g_{ul}^0]\phi_0$. Since $\langle \psi_0, B^0 \rangle = b \neq 0$, $\phi_0 \in N(g_u^0)$, we have $B^0 \notin R(g_u^0)$, $N(g_u^0) \cap N(\langle \phi_0, \cdot \rangle) = \{0\}$. Keller lemma [21,22] ensures that A^0 is nonsingular. The implicit function theorem leads that

$$\begin{cases} F(w, \eta, \varepsilon) = 0, \\ N(w, \eta, \varepsilon) = 0 \end{cases} \quad (6.3)$$

have solutions $(w(\varepsilon), \eta(\varepsilon))$, $\forall |\varepsilon| < \varepsilon_0$, which can be solved with the Newton iteration method. Therefore, we obtain the Σ symmetric positive solution branch $(u_s(l_0 + \eta(\varepsilon)) + \varepsilon(\phi_0 + w(\varepsilon)), l_0 + \eta(\varepsilon))$ of problem (1.1) which is switched from the $O(2)$ symmetric positive solution branch.

Remark 1. During actual computation, we need not calculate the second derivative in B^0 , because we always take $\varepsilon \neq 0$. Problem (6.3) is the following form ($q = 3$ here):

$$\begin{cases} \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 \phi_0}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi_0}{\partial \theta^2} \\ \quad + r^{(l_0+\eta)} (3u_s^2(l_0 + \eta)(\phi_0 + w) + 3\varepsilon u_s(l_0 + \eta)(\phi_0 + w)^2 + \varepsilon^2(\phi_0 + w)^3) = 0, \\ w|_{\partial\Omega} = 0, \\ \langle \phi_0, w \rangle = 0. \end{cases} \quad (6.4)$$

For given ε , we can obtain the solutions $(u_s(l_0 + \eta(\varepsilon)) + \varepsilon(\phi_0 + w(\varepsilon)), l_0 + \eta(\varepsilon))$ to (6.4) with the Newton iteration. Take ε from small to big enough, we can get the Σ symmetric positive solutions which can be continued by l continuation in X^Σ . (See Fig. 2).

7. Numerical results

Fig. 1 shows the $O(2)$ symmetric positive solution branch to problem (1.1). Fig. 2 shows the other different symmetric positive solutions bifurcated from the symmetry breaking bifurcation points. In Fig. 2 the $\Sigma_1, \Sigma_d, D_3, D_4, D_5, D_6, D_7$ and $O(2)$ symmetric positive solution branch are respectively drawn with different number from 1 to 8. When $0 < l < 0.59882964$, problem (1.1) has an $O(2)$ symmetric positive solution. When $0.59882964 < l < 3.19356079$, problem (1.1) has two positive solutions, which are respectively $O(2)$ and Σ_1 symmetric. When $3.19356079 < l < 5.77864075$, problem (1.1) has three positive solutions, which are respectively $O(2)$, Σ_1 and Σ_d symmetric. When $5.77864075 < l < 8.34997253$,

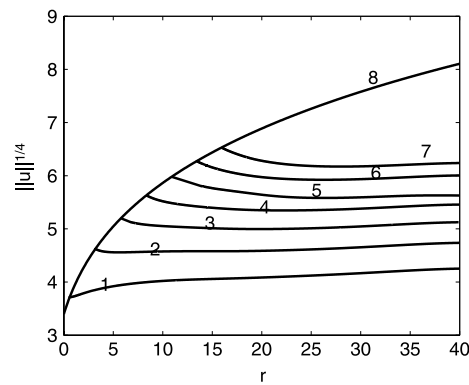


Fig. 2. Symmetry-breaking bifurcation diagram.

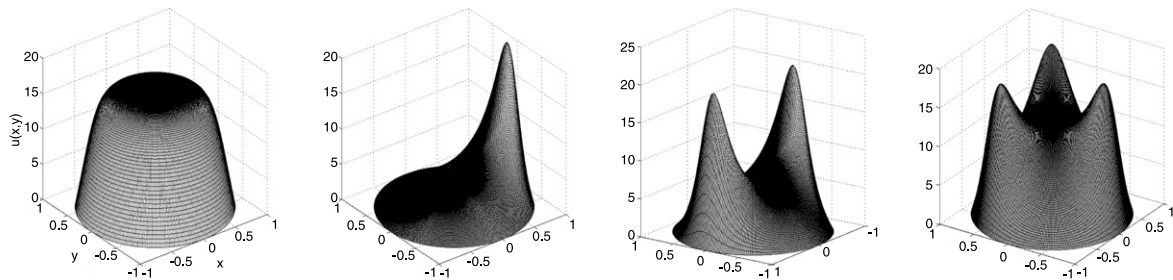


Fig. 3. Four positive solutions to (1.1) for $q = 3, l = 7$ with $O(2)$, Σ_1 , Σ_d and D_3 symmetry.

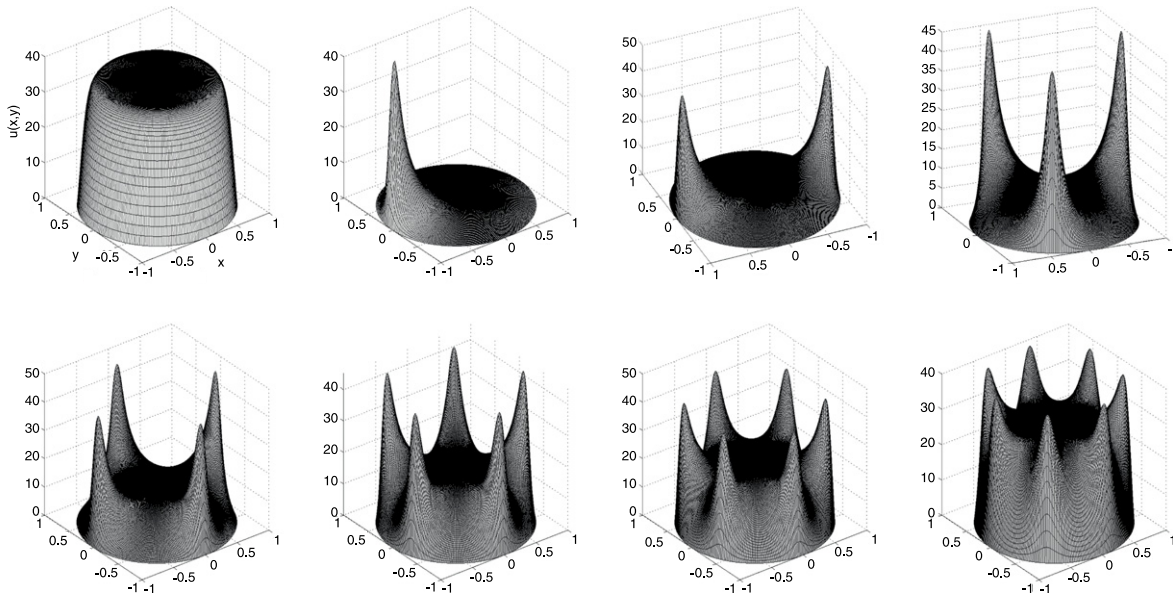


Fig. 4. Eight positive solutions to (1.1) for $q = 3, l = 18$ with $O(2)$, Σ_1 , Σ_d , D_3 , D_4 , D_5 , D_6 and D_7 symmetry.

problem (1.1) has four positive solutions, which are respectively $O(2)$, Σ_1 , Σ_d and D_3 symmetric. When $8.34997253 < l < 10.90316772$, problem (1.1) has five positive solutions, which are respectively $O(2)$, Σ_1 , Σ_d , D_3 and D_4 symmetric. When $10.90316772 < l < 13.43396789$, problem (1.1) has six positive solutions, which are respectively $O(2)$, Σ_1 , Σ_d , D_3 , D_4 and D_5 symmetric. When $13.43396789 < l < 15.93812666$, problem (1.1) has seven positive solutions, which are respectively $O(2)$, Σ_1 , Σ_d , D_3 , D_4 , D_5 and D_6 symmetric. When $l > 15.93812666$, problem (1.1) has eight positive solutions, which are respectively $O(2)$, Σ_1 , Σ_d , D_3 , D_4 , D_5 , D_6 and D_7 symmetric. Four positive solutions to problem (1.1) for $q = 3, l = 7$ are visualized in Fig. 3. They are respectively $O(2)$, Σ_1 , Σ_d and D_3 symmetric. Eight positive solutions to problem (1.1) for $q = 3, l = 18$ are visualized in Fig. 4. They are respectively $O(2)$, Σ_1 , Σ_d , D_3 , D_4 , D_5 , D_6 , D_7 symmetric.

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